



# The entropic boundary law in BF theory

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## Abstract

We compute the entropy of a closed bounded region of space for pure 3d Riemannian gravity formulated as a topological BF theory for the gauge group  $SU(2)$  and show its holographic behavior. More precisely, we consider a fixed graph embedded in space and study the flat connection spin network state without and with particle-like topological defects. We regularize and compute exactly the entanglement for a bipartite splitting of the graph and show it scales at leading order with the number of vertices on the boundary (or equivalently with the number of loops crossing the boundary). More generally these results apply to BF theory with any compact gauge group in any space–time dimension.

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## 1. Introduction

Entropy is a key notion in the search for a quantum gravity, both as the thermodynamic quantity useful to probe the physics and potential phenomenology of the theory and as the measure of information useful to identify the physical degrees of freedom and their dynamics. Research has focused on the particular case of black holes and has lead to the holographic principle directly relating geometric quantities (the area) to the entropy in quantum gravity.

In the context of Loop Quantum Gravity (see [2] for reviews), most of the black hole entropy calculations have been performed in the framework of isolated horizons following the seminal work by Ashtekar, Baez and Krasnov [3]. Assuming precise boundary conditions for the gravitational fields on the horizon, they count the number of (kinematical) boundary states consistent with fixing the value of the area. Here we would like to get rid of the classical boundary: instead

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of considering a spin network state with a boundary specified classically, we look at a closed spin network and an arbitrary bipartite splitting into inside/outside regions (see e.g. [4]). The aim is to compute the entanglement between these two parts of the spin network for a physical quantum geometry state solving the Hamiltonian constraint. The first step that we take here is to work this out for BF theory instead of gravity. Indeed, BF theories are topological field theories lacking local degrees of freedom, thus allowing for an exact quantization (see e.g. [1]). In this context, we know the physical states solving all the constraints and can compute the entanglement explicitly. This turns out to be similar to ground state entanglement calculations in some spin models developed for topological quantum computation [5,6].

The motivation to analyze BF theory is that it is very close to gravity. First, in three space–time dimensions, 3d gravity is actually a topological BF theory with the Lorentz group as gauge group. Second, in four space–time dimensions, gravity can be reformulated as a constrained BF theory and we can work on a quantization scheme with quantum BF theory as the starting point. Of course, the fact that gravity has local degrees of freedom should matter in the end. Nevertheless, studying BF theory should allow us to develop mathematical tools and procedures later useful for loop gravity.

In the present paper, we start with a quick overview of BF theory and the definition of the physical quantum states as spin network states for the flat connection with possibly particle-like topological defects. Then focusing on  $SU(2)$  BF theory, we explicitly compute the entanglement between the two parts of such a flat spin network states and we show its holographic behavior: it scales with the size of the boundary (more precisely, with the number of boundary vertices). We also study the influence of topological defects. We show that they do only affect the entropy when located on the boundary between the two regions and we compute the finite variation of entanglement that they create.

## 2. An overview of BF theory

### 2.1. $SU(2)$ BF theory: Spin networks and physical states

BF theory is a class of topological gauge field theories defined on a oriented smooth  $n$ -dimensional manifold  $\mathcal{M}$  by the following action (see e.g. [1] for a review):

$$S[A, B] = \int_{\mathcal{M}} \text{tr}(B \wedge F[A]). \quad (1)$$

The gauge group is a (semi-simple compact) Lie group  $G$  whose Lie algebra  $\mathfrak{g}$  is equipped with an invariant (non-degenerate) bilinear form  $\text{tr}(\cdot, \cdot)$ . Picking a local trivialization of the principal  $G$ -bundle over  $\mathcal{M}$ , the basic fields are a  $\mathfrak{g}$ -valued connection 1-form  $A$  with curvature 2-form  $F[A]$  and a  $\mathfrak{g}$ -valued  $(n-2)$ -form  $B$ . The action is invariant under the action of the gauge group, for  $h \in G$ :

$$B \rightarrow h B h^{-1}, \quad A \rightarrow h A h^{-1} + h d h^{-1}. \quad (2)$$

It is also invariant under shifts of the  $B$  field by an arbitrary  $(n-3)$ -form  $\phi$ :

$$B \rightarrow B + d_A \phi, \quad \delta A = 0. \quad (3)$$

This symmetry kills all local degree of freedom making BF theory a topological field theory. Its classical field equation impose a flat connection,  $F[A] = 0$ , and a vanishing “torsion”,  $d_A B = 0$ .

In particular, in  $n = 3$  space–time dimensions, BF theory for the gauge groups  $G = \text{SU}(2)$  and  $G = \text{SU}(1, 1)$  is equivalent to 3d Riemannian and Lorentzian gravity (in its first order formulation).

For our present purpose, we are interested into BF theory from the point of view of its canonical quantization à la loop quantum gravity. In the following, we will work with  $n = 3$  and  $G = \text{SU}(2)$  although all the formalism and techniques apply to arbitrary space–time dimensions and arbitrary (compact semi-simple) Lie groups. We perform a  $2 + 1$  splitting of the 3d space–time viewing the 3d manifold  $\mathcal{M} \sim \Sigma \times \mathbb{R}$  as a two-dimensional space  $\Sigma$  evolving in time. Then the spatial parts of the connection  $A$  and the field  $B$  on  $\Sigma$  are canonically conjugate variables. The theory is completely constrained and the Hamiltonian vanishes on-shell. There are two sets of constraints, imposed by the time components of  $A$  and  $B$ , respectively, which turn out to be simply Lagrange multipliers. The first constraints impose the flatness of the connection on  $\Sigma$ . The second set of constraints imposes the vanishing of the torsion on  $\Sigma$  and is usually called the *Gauss law*. These are first class constraints, respectively generating the translational symmetry (3) and the gauge invariance (2) under the action of the group  $G$ . For more details on the canonical analysis and resulting structures, the interested reader can check [2].

The loop quantization scheme is based on a specific choice of wave functions. We choose cylindrical functionals of the connection  $A$  on  $\Sigma$ . More precisely, they depend on  $A$  through only a finite number of variables: they are functions of the holonomies of  $A$  along the edges of some arbitrary (finite) oriented graph in  $\Sigma$ . Considering a particular graph  $\Gamma$  with  $E$  and  $V$  vertices, we consider the holonomies  $g_1[A], \dots, g_E[A]$  of the connection along the edges  $e = 1, \dots, E$  of  $\Gamma$  and build wave functions of the following type:

$$\psi_\Gamma(A) = \psi(g_1[A], \dots, g_E[A]).$$

Further we require these functionals to be gauge-invariant. Since the action of the group on the connection translates into a group action at the end points of the holonomies,  $g_e[A] \rightarrow h_{s(e)}^{-1} g_e[A] h_{t(e)}$ , where  $s(e)$  and  $t(e)$  are respectively the source and target vertices of the edge  $e$ , the gauge invariance of the wave functions  $\psi_\Gamma$  involves a invariance under the group action at every vertex of the graph  $\Gamma$ :

$$\forall h_v \in G^{\times V}, \quad \psi(g_1, \dots, g_E) = \psi(h_{s(1)}^{-1} g_1 h_{t(1)}, \dots, h_{s(E)}^{-1} g_E h_{t(E)}). \quad (4)$$

Then we need to impose the flatness condition  $F = 0$  on these wave functions. To keep the discussion as simple as possible, we consider a graph  $\Gamma$  which forms a lattice faithfully representing the canonical surface  $\Sigma$ . More precisely, if we cut the surface  $\Sigma$  along the embedded graph  $\Gamma$ , then we are left with surfaces all homomorphic to the unit disk. These are the faces of the lattice. We now impose that the holonomy of the connection around each face is the identity. Thus the projection onto physical states satisfying the flatness condition is implemented by the multiplication by  $\delta$ -function around each face:

$$\psi(g_e) \mapsto \prod_{\mathcal{L}} \delta\left(\prod_{e \in \mathcal{L}} g_e\right) \psi(g_e), \quad (5)$$

where  $\mathcal{L}$  labels the loops around the faces of the lattice. This imposes trivial holonomies around every contractible loop in  $\Sigma$  while allowing for arbitrary holonomies around the non-contractible cycles of the surface.

When  $\Sigma$  is the two-sphere, this gives a single flat quantum state. When  $\Sigma$  is an orientable compact surface of genus  $n$ , the physical space of flat quantum states is isomorphic to the space

of  $L^2$  functions of  $2n$  group elements  $A_1, B_1, \dots, A_n, B_n$  satisfying  $A_1 B_1 A_1^{-1} B_1^{-1} \dots A_n B_n A_n^{-1} B_n^{-1} = \mathbb{1}$  and invariant under diagonal conjugation  $A_i, B_i \mapsto h A_i h^{-1}, h B_i h^{-1}$  for  $h \in G$ . For instance, for the two-torus, physical states will be gauge invariant functions of two group elements  $A$  and  $B$  satisfying  $ABA^{-1}B^{-1} = \mathbb{1}$ .

Finally, in  $2+1$  space–time dimensions particles create conical singularities, which leave the space–time flat except along their worldline. They are represented as topological defects, which translate into non-trivial holonomies. For a spinless particle on a given face of the lattice, we replace the  $\delta$ -function by a  $\delta_\theta$ -function imposing that the holonomy around the face has a class angle  $\theta$ . This deficit angle  $\theta$  then defines the mass of the particle (e.g. see [7,8] for more details).

Our goal in the present work is to compute the entanglement on a physical state between a region of  $\Sigma$  and the rest of the surface and to check whether it satisfies to an “area–entropy” law. On a given graph  $\Gamma$ , we will consider a connected region of the graph and compute the von Neumann entropy of the reduced density matrix obtained after tracing out all the holonomies outside that region from a given physical state. In the next section, we will show how this works for a BF theory based on a discrete group, in which case the theory can be reformulated as a spin system.

To conclude this review section, we give a few mathematical details on the  $SU(2)$  group. We parameterize group elements  $g$  as

$$g = e^{i\theta \hat{u} \cdot \vec{\sigma}} = \cos \theta \mathbb{1} + i \sin \theta \hat{u} \cdot \vec{\sigma}, \quad (6)$$

where  $\theta \in [0, 2\pi]$  is the class angle (or half of the rotation angle),  $\hat{u}$  is a unit vector on the 2-sphere indicating the axis of the rotation and  $\vec{\sigma}$  are the standard Pauli matrices. Let us point out  $g(\theta, \hat{u}) = g(-\theta, -\hat{u})$ . In these variables the normalized Haar measure  $dg$  on  $SU(2)$  reads as

$$\int_{SU(2)} dg f(g) = \frac{1}{2\pi^2} \int_0^\pi \sin^2 \theta d\theta \int_{S^2} d^2 \hat{u} f(\theta, \hat{u}). \quad (7)$$

By the Peter–Weyl theorem, every function which is invariant under conjugation can be decomposed on the characters of the irreducible (spin) representations of  $SU(2)$ . Such representations are labeled by a half-integer  $j \in \mathbb{N}/2$  and the corresponding characters are

$$\chi_j(g) = \text{tr}_j(D^j(g)) = \frac{\sin(2j+1)\theta}{\sin \theta} = U_{2j}(\cos \theta), \quad (8)$$

where  $U_n$  is the  $n$ th Tchebyshev polynomial of the second kind. The  $\delta$ -distribution decomposes as:

$$\delta(g) = \sum_{j \in \mathbb{N}/2} (2j+1) \chi_j(g).$$

Finally, we introduce the distributions  $\delta_\theta$  that localize group elements on a specific equivalence class under conjugation by fixing their rotation angle:

$$\int dg \delta_\theta(g) f(g) = \frac{1}{4\pi} \int_{S^2} d^2 \hat{u} f((\theta, \hat{u})). \quad (9)$$

Its decomposition into characters reads:

$$\delta_\theta(g) = \sum_{j \in \mathbb{N}/2} \chi_j(\theta) \chi_j(g). \quad (10)$$

## 2.2. A word on Kitaev's spin system and $\mathbb{Z}_2$ BF theory

In this subsection, we give a quick overview of Kitaev's spin system introduced in [5] and the corresponding ground state entanglement calculations presented in [6]. As in the previous section, we consider the (canonical) surface  $\Sigma$  provided with the lattice defined by the graph  $\Gamma$ . We attach a two-level system (which is called a qubit in the language of quantum information and represents a spin-1/2 particle) to each edge  $e$  of the graph. Label its basis states as  $|\pm_e\rangle$ . Consider the following Hamiltonian which is a sum of local operators attached to the vertices and the faces of the lattice:

$$H = - \sum_v \bigotimes_{e \ni v} \sigma_x^{(e)} - \sum_f \bigotimes_{e \in f} \sigma_z^{(e)}, \quad (11)$$

with  $\sigma_x|\pm\rangle = |\mp\rangle$  and  $\sigma_z|\pm\rangle = \pm|\pm\rangle$ . Calling the operators  $\mathbf{A}_v \equiv \bigotimes_{e \ni v} \sigma_x^{(e)}$  and  $\mathbf{B}_f \equiv \bigotimes_{e \in f} \sigma_z^{(e)}$ , we first notice that all these operators commute with each other. We can thus diagonalize them simultaneously. Ground states  $|\psi_0\rangle$  are then states which diagonalize all  $\mathbf{A}_v$  and  $\mathbf{B}_f$  with the highest eigenvalue:

$$\mathbf{A}_v|\psi_0\rangle = \mathbf{B}_f|\psi_0\rangle = |\psi_0\rangle. \quad (12)$$

As shown in [5] (and reviewed in [6]), this spin system is equivalent to BF theory based on the discrete gauge group  $\mathbb{Z}_2$ . Generalizing this to systems with a higher number of levels attached to each edge allows to reformulate in a similar fashion BF theory for an arbitrary discrete group.

Ground states correspond to physical states in BF theory. The holonomy can either be '+' or '-' along an edge. Then the  $\mathbf{A}_v|\psi_0\rangle = |\psi_0\rangle$  condition implements the Gauss law imposing gauge invariance at each vertex, while the face condition  $\mathbf{B}_f|\psi_0\rangle = |\psi_0\rangle$  imposes the flatness of the  $\mathbb{Z}_2$  holonomy. Thus the Hilbert space of ground states for an orientable surface of genus  $g$  has a  $2^{2g}$  degeneracy, with the holonomy around all  $2g$  cycles of the surfaces being free.

Finally, using the stabilizer space methods, the entropy of one (bounded connected) region of the lattice was computed in any ground state [6]. It was shown that it does not depend on the particular ground state (i.e. non-contractible cycles do not matter) and it is simply related to the perimeter of the region's boundary as

$$S = n_L - 1,$$

where  $n_L$  is the number of spins in the perimeter of a region (for a 2d square lattice).

In the present work, we generalize these entanglement calculation to the case of the continuous Lie group, making explicit calculations in the case of the group  $G = \text{SU}(2)$ . We do not use the same methods as developed in the analysis of the spin systems but exploit simple loop quantum gravity tools.

## 3. Entanglement: Generic structures and entropy-boundary law

### 3.1. The setting: Flat spin networks and choice of independent loops

Let us consider a fixed connected oriented (abstract) graph  $\Gamma$  and a spin network wave functional  $\psi(g_e)$  living on it. We would like to study the completely flat wave functional, imposing that the holonomy along all loops of the graph. This corresponds to the flat physical state for BF theory for a trivial topology (a two-sphere) of the canonical surface  $\Sigma$ . It also gives the physical

state for a non-trivial topology but with the additional constraint of trivial holonomies around every cycle of the canonical surface. To truly consider non-trivial topologies, we would need to allow for non-trivial holonomies around some of the loops of the graph (which would then represent the non-contractible cycles of the canonical surface).

In order to study the entanglement properties of this completely flat state, we write the flat wave functional  $\psi_0(g_e)$  as a product of  $\delta$ -functions on the group  $SU(2)$  for every loop of the graph. However, such a naive product would obviously lead to infinities due to redundant  $\delta$ -functions and we need to consider the product over independent loops. We formalize this using the gauge fixing procedure for spin networks that was developed in [11].

Considering a generic spin network functional  $\psi(g_e)$  on  $\Gamma$ , we can gauge fix the  $SU(2)$  gauge transformations (4) acting at the vertices by introducing a maximal tree  $T$  on the graph.  $T$  is a connected set of edges on  $\Gamma$  touching all vertices of  $\Gamma$  but never forming any loop. If we call  $E$  and  $V$  respectively the number of edges and vertices of  $\Gamma$ , then the number of edges in  $T$  is exactly  $V - 1$ . On the other hand, the number of edges not in  $T$  is exactly the number of independent loops on the graph  $L \equiv E - V + 1$ . The gauge fixing consists in fixing all the group elements  $g_e$  living on edges  $e \in T$  belonging to the tree to the identity. More precisely, we choose a reference vertex  $v_0$ . Then for all vertices  $v$ , there exists a unique path  $[v_0 \rightarrow v]$  linking  $v_0$  to  $v$  along the tree  $T$ . We perform a gauge transformation with parameters:

$$h_v \equiv \overrightarrow{\prod}_{e \in [v_0 \rightarrow v]} g_e.$$

For all edges on the tree  $e \in T$ , the resulting group elements  $h_{s(e)}g_e h_{t(e)}^{-1}$  are set to the identity  $\mathbb{1}$ . For all edges not belonging to the tree  $e \notin T$ , this defines a loop variable  $G_e \equiv h_{s(e)}g_e h_{t(e)}^{-1}$ , which is the oriented product of the loop  $\mathcal{L}_e \equiv [v_0 \rightarrow s(e)] \cup e \cup [t(e) \rightarrow v_0]$  starting at  $v_0$  and going to  $s(e)$  along the tree  $T$ , and then coming back to  $v_0$  along the tree from the vertex  $t(e)$ . The procedure ensures that (see [11] for more details) the wave functions evaluated on the original  $g_e$ 's is equal to its evaluation on the  $G_{e \notin T}$  while setting the other group elements to  $\mathbb{1}$ .

Finally, we define the completely flat spin network state as:

$$\psi_0(g_e) \equiv \prod_{e \notin T} \delta(G_e). \quad (13)$$

It is straightforward to check that the resulting state actually does not depend on the choice of neither the reference vertex  $v_0$  nor the maximal tree  $T$ . Moreover, it truly imposes the condition that the holonomy around any loop of the graph  $\Gamma$  is constrained to be equal to the identity  $\mathbb{1}$ .

Such a distributional state is obviously not normalisable for the kinematic inner product defined with the Haar measure  $\int dG_e$ . It is indeed  $L^1$  but not  $L^2$ . To deal with it, we need to regularize it. The method we will use is the standard one when working in loop quantum gravity and spin foam models [2]: expanding the state in  $SU(2)$  representations, we will introduce by hand a cut-off  $J$  in the representations and then study the behavior of the various quantities in the large spin limit  $J \rightarrow +\infty$ .

Our purpose is to consider a bounded region  $A$  of the graph  $\Gamma$  and compute the entanglement between  $A$  and the rest of the graph on the completely flat spin network state. This will give the entropy of  $A$ . More precisely, we choose a connected region  $A$  of  $\Gamma$ . We define it as a set of  $V_{\text{int}}$  vertices and the  $E_{\text{int}}$  edges that link them (i.e. any edge whose both source and target vertices are in  $A$  also belongs to  $A$ ). We further distinguish the  $E_b$  boundary edges, that connect one vertex inside  $A$  to one vertex outside, and the  $E_{\text{ext}}$  exterior edges who do not touch the

considered region  $A$ . We call the interior graph  $\Gamma_{\text{int}}$  the graph formed by the vertices and (interior) edges of  $A$ . The exterior graph  $\Gamma_{\text{ext}}$  is defined as its complement: it consists in both exterior and boundary edges. We define the  $V_b$  boundary vertices which belong to both interior and exterior graphs i.e. vertices in  $A$  that touch some boundary edges. Then the exterior graph has  $V_{\text{ext}} = V - V_{\text{int}} + V_b$  vertices.

To define the entanglement between the interior and the exterior regions, we consider the reduced density matrix on  $A$  obtained by tracing out all the holonomies outside  $A$  from the full density matrix:

$$\begin{aligned} \rho(g_e, \tilde{g}_e) &\equiv \psi(g_e) \bar{\psi}(\tilde{g}_e), \\ \rho_{\text{int}}(g_{e \in A}, \tilde{g}_{e \in A}) &= \int [dg_{e \notin A}] \psi(g_{e \in A}, g_{e \notin A}) \bar{\psi}(\tilde{g}_{e \in A}, g_{e \notin A}). \end{aligned} \quad (14)$$

We are then interested in the standard measure of entanglement defined as  $E = -\text{tr}_A \rho_{\text{int}} \log \rho_{\text{int}}$ . We would like to compute it on the completely flat spin network state, which a physical state for BF theory. For this purpose, we need to adapt the choice of the tree  $T$  used in defining the flat state  $\psi_0$  to the choice of the studied region  $A$ : we would like the definition of  $\psi_0$  to respect the interior/boundary/exterior structure.

We now choose two reference vertices  $v_0$  in the interior (there is no problem if  $v_0$  itself is a boundary vertex) and  $w_0$  in the exterior (for the sake of simplicity, we choose  $w_0$  so that it is not a boundary vertex) and two maximal trees  $T_{\text{int}}$ ,  $T_{\text{ext}}$  respectively on the interior and outer graphs. We would like to form a maximal tree  $T$  on the whole graph by merging the two trees. The only issue is that the interior and outer graphs, and thus the two trees, both share the  $V_b$  boundary vertices. Considering a boundary vertex  $v_b$ , there exists a unique path along the tree  $T_{\text{int}}$  from  $v_0$  to  $v_b$  and there also exists a unique path in the exterior along  $T_{\text{ext}}$  from  $w_0$  to  $v_b$ . If we consider the straightforward gluing of the two trees  $T_{\text{int}} \cup T_{\text{ext}}$ , then we obtain loops as soon as there are at least two boundary vertices of the type  $[v_0 \rightarrow v_b^{(1)} \rightarrow w_0 \rightarrow v_b^{(2)} \rightarrow v_0]$ . In order to get a tree, we simply need to break these loops. For this purpose, we single out an arbitrary boundary vertex  $v_b^{(0)}$  and we number the other boundary vertices  $v_b^{(i)}$  with  $i = 1, \dots, (V_b - 1)$ . We will remove one edge from each loop  $[v_0 \rightarrow v_b^{(0)} \rightarrow w_0 \rightarrow v_b^{(i)} \rightarrow v_0]$  along  $T_{\text{int}} \cup T_{\text{ext}}$ . More precisely, for each  $v_b^{(i)}$ , we consider the unique boundary edge  $e^{(i)}$  in  $T_{\text{ext}}$  touching  $v_b^{(i)}$ . Finally, we define  $T \equiv (T_{\text{int}} \cup T_{\text{ext}}) \setminus \{e^{(i)}\}$  and it is straightforward to check that  $T$  is a maximal tree on the whole graph  $\Gamma$ . Indeed there are no loops in  $T$ . Moreover, there exists a path in  $T$  from  $v_0$  to any vertex  $v \in \Gamma$ : if  $v$  is in the region  $A$  the path is simply the path  $[v_0 \rightarrow v]$  within  $T_{\text{int}}$  while if  $v$  is outside  $A$  we consider the sequence of edges  $[v_0 \rightarrow v_b^{(0)} \rightarrow w] \cup [w \rightarrow v]$  with the first halves of the path in  $T_{\text{int}}$  and the second in  $T_{\text{ext}}$ .

We follow the previous gauge fixing procedure and define the flat state  $\psi_0$  according to this chosen maximal  $T$ . We have one holonomy loop variable per edge  $e \notin T$  not belonging to the tree  $T$ . There are three types of such edges. First, we identify the edges  $e$  in the region  $A$  but not belonging to the interior tree  $T_{\text{int}}$ . Second, we identify the edges  $e$  outside  $A$  but not belonging to the outer tree  $T_{\text{ext}}$ . Finally, there are the boundary edges  $e^{(i)}$ ,  $i = 1, \dots, (V_b - 1)$ . For all these edges  $e \notin T$ , we associate the corresponding holonomies around the interior, exterior and boundary loops respectively which consist in edges  $f$  going from  $v_0$  to  $s(e)$  along  $T$  then along the edge  $e$  then coming back from  $t(e)$  to  $v_0$  along  $T$ . We call the loops  $\mathcal{L}_e \equiv [v_0 \rightarrow s(e)] \cup e \cup [t(e) \rightarrow v_0]$  and define the corresponding holonomies:

$$\forall e \in \Gamma_{\text{int}} \setminus T_{\text{int}}, \quad G_e \equiv \prod_{f \in \mathcal{L}_e}^{\rightarrow} g_f, \quad \forall e \in \Gamma_{\text{ext}} \setminus T_{\text{ext}}, \quad H_e \equiv \prod_{f \in \mathcal{L}_e}^{\rightarrow} g_f,$$

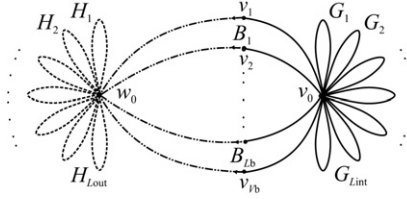


Fig. 1. The interior and the exterior flowers are linked by  $V_b$  boundary edges. Interior edges are shown as solid lines.

$$\forall i = 1, \dots, (V_b - 1), \quad B_i \equiv \prod_{f \in \mathcal{L}_e(i)} g_f. \quad (15)$$

The flat state is then the product of  $\delta$ -functions over all these loops:

$$\psi_0(g_e) \equiv \prod_{e \in \Gamma_{\text{int}} \setminus T_{\text{int}}} \delta(G_e) \prod_{e \in \Gamma_{\text{ext}} \setminus T_{\text{ext}}} \delta(H_e) \prod_{i=1 \dots (V_b-1)} \delta(B_i). \quad (16)$$

We insist that the whole procedure with the choice of a tree  $T$  is simply to choose a set of independent loops in order to impose the flatness conditions with no redundant  $\delta$ -distribution. In other words, the state  $\psi_0$  as defined above still imposes that the holonomy around any loop of the whole graph  $\Gamma$  is trivial.

Finally, a last detail is that we can cut the loops  $\mathcal{L}_e$  for exterior edges  $e \in \Gamma_{\text{ext}} \setminus T_{\text{ext}}$ . We can have them start at the exterior vertex  $w_0$  instead of the reference vertex  $v_0$ : we define  $\tilde{\mathcal{L}}_e \equiv [w_0 \rightarrow s(e)] \cup e \cup [t(e) \rightarrow w_0]$ . This leads to the holonomies  $H^{-1} H_e H$  where  $H$  is the oriented product of the group elements from  $v_0$  to  $w_0$  along the tree  $T$  (going through  $v_b^{(0)}$ ). Since the  $\delta$ -functions are central, replacing the  $H_e$ 's by  $H^{-1} H_e H$  does not change anything to the definition of the flat state  $\psi_0$ . On the other hand, the loops  $\tilde{\mathcal{L}}_e$  have the advantage that they only involve edges in the outer graph  $\Gamma_{\text{ext}}$  (i.e. not belonging to the region  $A$ ). We will therefore use this prescription for the entanglement calculations (see Fig. 1).

We would like to underline that the properties (entanglement, ...) of the flat state  $\psi_0$  do not depend on the specific choice of tree  $T$  that we made. Indeed the choice of tree is simply a choice to gauge fixing for the spin network functional. The particular tree that we defined using trees in the interior and exterior regions is a convenient choice allowing a clear description of the “boundary loops” which are the central objects for the entanglement calculation.

Readers uninterested in the details of the derivations can skip to the statement of [Result 1](#), the definition of Eq. (25), [Results 2 and 3](#), and to [Section 3](#) that presents some simple examples. On the other hand, a more detailed proof of the following results can be found in [\[17\]](#).

### 3.2. Computing the entanglement for the flat state

We now compute the reduced density matrix  $\rho_{\text{int}}$  obtained by tracing out all exterior holonomies from the flat state  $\psi_0$ . The von Neumann entropy of this reduced density matrix defines the entanglement between the region  $A$  and the rest of the spin network. Our main result is that the entanglement scales with the size of the boundary:

$$E_A \equiv S[\rho_{\text{int}}] \equiv -\text{tr}_A \rho_{\text{int}} \log \rho_{\text{int}} = (V_b - 1) u(J), \quad (17)$$

where  $(V_b - 1)$  is the number of boundary loops between the region  $A$  and the exterior and  $u(J)$  is a unit of entanglement which only depends on the regulator  $J$  (representation cut-off) and does



not depend on either the graph  $\Gamma$  or the choice of the region  $A$ . In particular, if there is a single boundary vertex,  $V_b = 1$ , the region  $A$  is totally disentangled from the rest of the spin network.

Let us compute the reduced density matrix as defined by (14):

$$\rho_{\text{int}}(g_e, \tilde{g}_e) = \int [dg_{e \notin A} d\tilde{g}_{e \notin A}] \prod_{e \notin A} \delta(\tilde{g}_e g_e^{-1}) \prod_{e \notin T} \delta(G_e) \delta(B_i) \delta(H_e) \prod_{e \notin T} \delta(\tilde{G}_e) \delta(\tilde{B}_i) \delta(\tilde{H}_e).$$

First, the interior loops are unaffected by the integration over exterior edges since they do not involve any edges outside the region  $A$ . Second, the exterior loops involve only exterior edges, thus we have the identification  $\tilde{H}_e = H_e$ . This produces  $\delta(1)$  infinities which we re-absorb in the normalisation of the reduced density matrix. Finally, the moot point is what happens to the boundary loops. More precisely, we have defined:

$$B_i = C_0 D_0^{-1} D_i C_i^{-1}, \quad \text{with } C_i \equiv \prod_{f \in [v_0 \rightarrow v_b^{(i)}] \subset \Gamma_{\text{int}}} g_f, \quad D_i \equiv \prod_{f \in [w_0 \rightarrow v_b^{(i)}] \subset \Gamma_{\text{ext}}} g_f. \quad (18)$$

With this decomposition the identification  $\delta(\tilde{g}_e g_e^{-1})$  for all edges  $e \in \Gamma_{\text{ext}}$  implies that the factors  $\delta(B_i) \delta(\tilde{B}_i)$  can be re-written as  $\delta(C_i^{-1} C_0 \tilde{C}_0^{-1} \tilde{C}_i)$  where the holonomy  $C_i^{-1} C_0$  is the (oriented) product of the group elements from the boundary vertex  $v_b^{(i)}$  to  $v_b^{(0)}$  along the interior tree  $T_{\text{int}}$ . Therefore, the reduced density matrix reads up to a normalisation (later fixed by the requirement that  $\text{tr}_A \rho_{\text{int}} = 1$ ):

$$\rho_{\text{int}}(g_{e \in A}, \tilde{g}_{e \in A}) = \prod_{e \notin T_{\text{int}}} \delta(G_e) \delta(\tilde{G}_e) \prod_{i=1}^{(V_b-1)} \delta(C_i^{-1} C_0 \tilde{C}_0^{-1} \tilde{C}_i). \quad (19)$$

The next step is to compute the von Neumann entropy of this density matrix,  $S_A = -\text{tr}_A \rho_{\text{int}} \log \rho_{\text{int}}$ . We have two types of terms: some  $\delta(G_e) \delta(\tilde{G}_e)$  for edges not in the tree  $e \notin T_{\text{int}}$  and some  $\delta(g_i \tilde{g}_i^{-1})$  which only involve edges in the tree  $T_{\text{int}}$ . Since these different terms do not involve the same group elements, they can be treated separately (more precisely, we can do a change of variable in the integrations replacing the  $g_e$ 's by the  $G_e$ 's for the edges not in the tree  $T_{\text{int}}$  and the Jacobian of the transformation is trivial due to the left and right invariance of the Haar measure). The terms  $\delta(G_e) \delta(\tilde{G}_e)$  corresponding to the interior loops are density matrices corresponding to pure states and thus have a zero entropy. The only non-vanishing contribution therefore comes from the boundary loops and gives:

$$S_A = (V_b - 1) S[\delta(g \tilde{g}^{-1})], \quad (20)$$

where  $S[\delta(g \tilde{g}^{-1})]$  is the entropy of the to-be-normalized density matrix  $\sigma(g, \tilde{g}) = \delta(g \tilde{g}^{-1})$ . It turns out that this density matrix  $\sigma$  is actually the identity matrix and its entropy is simply the log of the dimension of the Hilbert space of  $L^2$  functions on  $\text{SU}(2)$ . However, this Hilbert space has an infinite dimension and the result requires a regularization.

The regularization consists in introducing a cut-off  $J$  in the representations of  $\text{SU}(2)$ . We define a regularized  $\delta_J$  function in term of the  $\text{SU}(2)$  characters<sup>1</sup>:

$$\delta_J(g \tilde{g}^{-1}) = \sum_{j \in \mathbb{N}/2}^{j \leq J} (2j+1) \chi_j(g \tilde{g}^{-1}). \quad (21)$$

<sup>1</sup> We could also use the  $q$ -deformation of  $\text{SU}(2)$  which provides a natural way to restrict to a finite number of representations [14]. For instance, given the deformation parameter  $q = \exp(i\pi/J)$ , the highest representation is  $j = J - 1/2$ .

Correspondingly, we consider the Hilbert space of  $L^2$  functions on  $SU(2)$  which decompose onto matrix elements of the group elements involving only representations  $j \leq J$ . This Hilbert space  $\mathcal{H}_J$  is spanned by the (renormalized) (Wigner) matrix elements  $\sqrt{2j+1} D_{mn}^j(g)$ , where  $m$  and  $n$  run by integer step from  $-j$  to  $j$ . Its dimension is thus:

$$\Delta_J \equiv \sum_{j \in \mathbb{N}/2}^{j \leq J} (2j+1)^2 = \frac{1}{3}(1+2J)(1+J)(3+4J) \sim 8J^3/3. \quad (22)$$

It is easy to check that the non-normalized density matrix  $\delta_J(g\tilde{g}^{-1})$  is the identity on  $\mathcal{H}_J$  by explicitly computing its matrix elements<sup>2</sup> and therefore its entropy is  $S(\delta_J) = \log \Delta_J$ . Indeed, if we introduce the normalized kets through

$$\langle g|j, m, n\rangle = \sqrt{2j+1} D^{(j)}(g)_n^m, \quad \langle j, m, n|j', m', n'\rangle = \delta_{jj'} \delta_{mm'} \delta_{nn'},$$

then the normalized density matrix

$$\rho^J = \frac{1}{\Delta_J} \sum_{j \in \mathbb{N}/2}^{j \leq J} \sum_{m,n} |j, m, n\rangle \langle j, m, n|, \quad (23)$$

is obviously maximally mixed, hence is maximally entangled with the rest of the system [15]. This finally proves the results which we announced above (a more detailed proof is presented in [Appendix A](#)):

**Result 1.** *The entanglement between the region  $A$  and the rest of the spin network state for the completely flat state  $\psi_0$  is:*

$$S_A = E_A[\psi_0] = (V_b - 1) \log \Delta_J, \quad \text{with } \log \Delta_J \underset{J \rightarrow \infty}{\sim} 3 \log J + \log \frac{8}{3}. \quad (24)$$

---

Their  $q$ -dimensions are equal to

$$d_j^{(q)} \equiv \frac{\sin(2j+1)\pi/J}{\sin \pi/J} = \chi_j(\pi/J),$$

which is the same as the evaluation of the usual  $SU(2)$  character on an angle  $\theta = \pi/J$ . When  $J \rightarrow \infty$  (and  $q \rightarrow 1$ ), we recover  $d_j^{(q)} \rightarrow (2j+1)$ . Hence the  $q$ -deformed version of the reduced density matrix  $\sigma(g, \tilde{g})$  is

$$\rho_{(q)}(g, \tilde{g}) = \sum_j d_j^{(q)} \chi_j(g\tilde{g}^{-1}).$$

Its entropy (after properly normalization of the density matrix) is

$$S(\rho_q) = \log \sum_j [d_j^{(q)}]^2 = \log N(J, \pi/J) \sim 3 \log J + \dots,$$

where the factor  $N(J, \pi/J)$  is introduced later when dealing with topological defects.

<sup>2</sup> We compute, using the orthogonality of  $SU(2)$  matrix elements:

$$\langle \sqrt{2k+1} D_{cd}^k | \delta_J | \sqrt{2j+1} D_{ab}^j \rangle = \int dg d\tilde{g} \sqrt{(2k+1)(2j+1)} D_{ab}^j(g) \overline{D_{cd}^k(\tilde{g})} \delta_J(g\tilde{g}^{-1}) = \delta_{jk} \delta_{ac} \delta_{bd}.$$

At the end of the day, it is the number of boundary vertices that is relevant for the entanglement. It could seem surprising since the  $SU(2)$  group elements live on the edges of the graph and therefore the degrees of freedom of the theory apparently live on the edges. We would then expect the entanglement to scale with the number boundary edges. However, the requirement of gauge invariance implies that the physical degrees of freedom actually live on the (independent) loops of the graph and not simply on its edges. Then we proved that the number of (independent) boundary loops is directly related to the number of boundary vertices (minus one). A last comment is that we would still have found the same leading order behavior for the entanglement in case we had first regularize and then calculate the reduced state instead of the contrary as we did above.<sup>3</sup>

### 3.3. Topological defects and renormalized entanglement

We can further generalize the procedure described above to take into account topological defects. In the context of 3d gravity, topological defects represent point particles. For instance, a (spinless) particle of given mass leads to a non-trivial holonomy around it: we impose  $\delta_\theta(g)$  instead of  $\delta(g)$  on loops around it with the angle  $\theta$  related to the mass (see [7,8] for more details).

Moreover, we expect that the leading order of the entanglement does not change when including topological defects. Nevertheless, for any state  $\psi$  we introduce the renormalized (or relative) entanglement as a regularized entropy [13] of the reduced density operator of  $A$ ,  $\tilde{E}_A[\psi] \equiv \tilde{S}(\rho_{\text{int}})$

$$\tilde{S}[\rho_{\text{int}}] \equiv \lim_{J \rightarrow \infty} S(\rho_{\text{int}}^J) - S(\rho_{0\text{int}}^J), \quad (25)$$

which is the difference of the regularized measure of entanglement of the region  $A$  with the rest computed in the spin network state  $\psi$  and the entanglement computed for the completely flat state  $\psi_0$ . When  $\psi$  only includes topological defects, we actually expect the renormalized entanglement to converge to a finite value as the regulator  $J$  is taken to infinity.

Let us first consider inserting a topological defect in an interior loop or an exterior loop,<sup>4</sup> i.e. replacing  $\delta(G_e)$  or  $\delta(H_e)$  by  $\delta_\theta(G_e)$  or  $\delta_\theta(H_e)$ . In the case of an exterior loop, we will get in the reduced density matrix after integration over the group elements living on the exterior edges a term  $\int dH \delta_\theta(H)^2$  instead of  $\int dH \delta(H)^2$ . Such a term only enters the normalization of the reduced density matrix and thus does not affect the entanglement. In the case of an interior loop, the modified loop constraint gives a term  $\delta_\theta(G_e)\delta_\theta(\tilde{G}_e)$  in the reduced density matrix instead of the original  $\delta(G_e)\delta(\tilde{G}_e)$ . This still represents a pure state, thus has zero entropy. Once again, it does not affect the entanglement. This proves the following result:

<sup>3</sup> It can be understood by noting that

$$\int dg \delta(g) \delta(g) = \delta(\mathbb{1}) = \sum_{j \in \mathbb{N}/2} d_j^2 \rightarrow \Delta_J,$$

and regularization of  $\delta$ -function prior to the integration produces the same result:

$$\sum_{j,k \in \mathbb{N}/2}^J d_j d_k \int dg \chi_j(g) \chi_k(g) = \sum_{j,k \in \mathbb{N}/2}^J d_j d_k \delta_{jk} = \Delta_J.$$

<sup>4</sup> We can further introduce any topological defect on arbitrary products of these interior and exterior loops of the type  $\delta_\theta(G_{e_1} G_{e_2} G_{e_3} \dots)$  and  $\delta_\theta(H_{e_1} H_{e_2} H_{e_3} \dots)$ . As shown in the detailed proof presented in Appendix A, this does not affect the entanglement at all, which turns out to depend only on the boundary loops.

**Result 2.** *Starting off from the completely flat state  $\psi_0$  and then adding topological defects along some loops of the spin network state, if no topological defect is inserted on boundary loops then the entanglement between the region  $A$  and the rest of the graph does not change:*

$$\bar{E}_A[\psi] = 0, \quad \text{or equivalently} \quad E_A[\psi] = (V_b - 1) \log \Delta_J. \quad (26)$$

On the other hand, if we insert a topological defect on the boundary between  $A$  and the outside, or more precisely replace the constraint  $\delta(B_i)$  by  $\delta_\theta(B_i)$  on one given boundary loop, this will modify our previous entanglement calculation. For the completely flat state  $\psi_0$ , the term in  $\rho_{\text{int}}$  corresponding to that loop was  $\int dg d\tilde{g} \delta(g\tilde{g}^{-1}) \delta(g_i g) \delta(\tilde{g}_i \tilde{g}) = \delta(g_i \tilde{g}_i^{-1})$  which gives the totally mixed state (identity density matrix on the Hilbert space of  $L^2$  functions over  $\text{SU}(2)$ ). This now replaced by:

$$\begin{aligned} \int dg d\tilde{g} \delta(g\tilde{g}^{-1}) \delta_\theta(g_i g) \delta_\theta(\tilde{g}_i \tilde{g}) &= \int dg \delta_\theta(g_i g) \delta_\theta(\tilde{g}_i g^{-1}) \\ &= \sum_j \frac{\chi_j(\theta)^2}{(2j+1)} \chi_j(g_i \tilde{g}_i^{-1}), \end{aligned} \quad (27)$$

where we used the decomposition of the distribution  $\delta_\theta$  into  $\text{SU}(2)$  representations as given by Eq. (10). Cutting off the sum over representations to  $J$ , we compute the matrix elements of this reduced density matrix:

$$\langle \sqrt{2k+1} D_{cd}^k | \sigma_J^{(\theta)} | \sqrt{2j+1} D_{ab}^j \rangle = \frac{1}{N(J, \theta)} \delta_{jk} \left( \frac{\chi_j(\theta)}{2j+1} \right)^2 \delta_{ac} \delta_{bd}, \quad (28)$$

where the normalisation factor  $N(J, \theta)$  ensures that the reduced density matrix  $\sigma_J^{(\theta)}$  has a unit trace:

$$N(J, \theta) = \sum_{j \leq J} \chi_j(\theta)^2 = \frac{(3+4J) \sin \theta - \sin(3+4J)\theta}{4 \sin^3 \theta}. \quad (29)$$

One can check that doing a Taylor expansion around  $\theta \sim 0$ , we recover  $N(J, \theta \rightarrow 0) = \Delta_J$ . Finally, computing the von Neuman entropy of this density matrix gives the renormalized entanglement between the region  $A$  and the outside in the state  $\psi_\theta$  with one topological insertion along a boundary loop:

**Result 3.** *Inserting a single topological defect  $\delta_\theta$  along a boundary loop between the region  $A$  and the outside leads to a non-zero renormalized entanglement:*

$$\bar{E}_A[\theta] = \lim_{J \rightarrow \infty} \log \frac{N(J, \theta)}{\Delta_J} - \frac{1}{N(J, \theta)} \sum_{j \leq J} \chi_j(\theta)^2 \log \frac{\chi_j(\theta)^2}{(2j+1)^2}. \quad (30)$$

We prove in Appendix B that this expression has a finite limit for all values of  $\theta$  when the regulator  $J$  is sent to  $\infty$ . This renormalized entanglement has a universal value when  $\theta$  is not a rational fraction of  $\pi$ :

$$\bar{E}_A(\theta) = -3 + \log 6 \approx -1.20824. \quad (31)$$

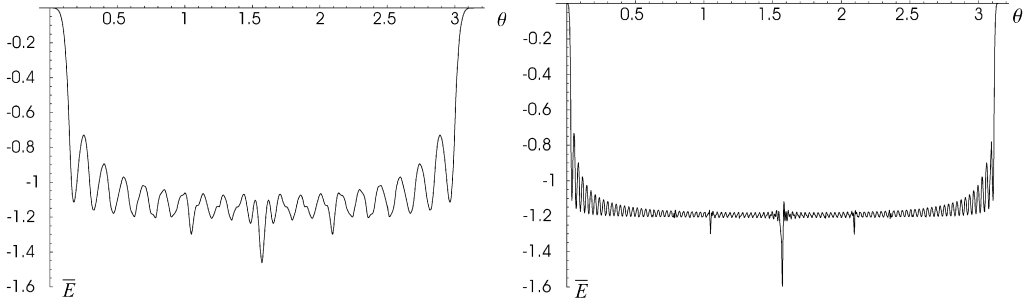


Fig. 2. The regularized entanglement  $\bar{E}^J[\theta] = E_A^J(\psi_\theta) - E_A^J(\psi_0)$  for different values of the representation cut-off  $J = 10$  and  $J = 50\frac{1}{2}$ .

We can also compute this value analytically<sup>5</sup> for some specific values of the angle  $\theta$ :

$$\bar{E}_A(\pi/2) = -2 + \log(3/2) \approx -1.5945,$$

$$\bar{E}_A(\pi/3) = -2 + \log 2 \approx -1.30436,$$

$$\bar{E}_A(\pi/4) = -2 + \log 3 - \frac{1}{2} \log 2 \approx -1.248.$$

We see that the renormalized entropy (and entanglement) is discontinuous. This is actually a generic feature of infinite-dimensional systems [9,12]. It is also known that if a reasonable constraint is put on the accessible set of states, such as bounded mean energy,  $\text{tr } H\rho < \infty$ , then the entropy will be a continuous function on this set. In our case the entropy is continuous if the cut-off  $J$  is large but finite. This can be motivated by appealing to the geometric interpretation of the representation labels. Indeed, in the context of  $2+1$  (loop) gravity, the spin network states are the eigenstates of the length operator and its eigenvalues are exactly the representation labels (up to ordering ambiguities). Thus requiring the finite extent of the system in space would naturally impose an upper cut-off on the edge representation labels. (See Fig. 2.)

The negativity of the above result can be easily understood if we recall the flat holonomy state is maximally entangled. Therefore, the states with non-trivial holonomies are less than maximally entangled, hence the negativity of the renormalized entropy.

This result extends to the more general case of several topological defects inserted along different boundary loops, each boundary loop contributing independently to the (renormalized) entanglement between the region  $A$  and the rest of the spin network state.

Finally, we are left with the possibility of a non-trivial (2d) topology, i.e. non-contractible cycles in the canonical surface. It is easy to see that the entanglement calculations are not affected by non-contractible cycles as long as the region  $A$  has a trivial topology (isomorphic to the unit disk). Otherwise, if the region  $A$  contains some non-contractible cycles, the holonomies around them will couple to the holonomies around the cycles outside  $A$ . As an example, taking the case of a 2-torus, let us consider that one cycle with holonomy  $G$  is contained inside  $A$  while the dual

<sup>5</sup> For example, for  $\theta = \pi/2$ ,  $\chi_j(\pi/2)$  vanishes for all half-integer values of  $j$  and is equal to  $(-1)^j$  when the representation label  $j$  is an integer. We can then compute the renormalized entanglement using the following exact sum:

$$\sum_{j \in \mathbb{N}}^{j \leq J} \log(2j+1) = -\frac{1}{2} \log \pi + \log(2^{J+1} \Gamma(J + \frac{3}{2})).$$

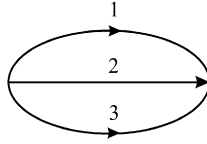


Fig. 3. The oriented  $\Theta$ -graph.

cycle with holonomy  $H$  is outside. Now, flatness of the connection does not require  $\delta(G)\delta(H)$  but the weaker condition that they commute  $GHG^{-1}H^{-1}$ . Then the reduced density matrix  $\rho_{\text{int}}$  for the region  $A$  will be given in term of integrals of the type:

$$\sigma(G, \tilde{G}) = \int dH \delta(GHG^{-1}H^{-1}) \delta(\tilde{G}H\tilde{G}^{-1}H^{-1}) f(H),$$

for some central function  $f$  (invariant under conjugation). The representation decomposition of such density matrix involves the  $\{6j\}$  symbol and are more complicated to analyze. We postpone this study to future investigation.

#### 4. Some simple examples of entanglement

##### 4.1. The $\Theta$ -graph

The simplest spin network that allows a non-trivial entropy calculation is the  $\Theta$ -graph. It is too simple to be decomposed into two non-trivial inside/outside regions, but it allows to illustrate how to compute the entanglement between two sets of holonomies. (See Fig. 3.)

The totally flat state is (the choice of maximal tree is a single edge  $T = \{e_3\}$ ):

$$\Psi(g_1, g_2, g_3) = \delta(g_1 g_3^{-1}) \delta(g_2 g_3^{-1}), \quad (32)$$

where we chose the two independent loops  $[e_1, e_3]$  and  $[e_2, e_3]$ . Notice that the  $\delta$ -functions also impose that the loop  $[e_1, e_2]$  carries a trivial holonomy,  $g_1 g_2^{-1} = \mathbb{1}$ . Constructing the (formal) density operator (up to regularisation/renormalisation) for the edge  $e_3$  one obtains

$$\begin{aligned} \rho(g_3, \tilde{g}_3) &= \int dg_1 dg_2 \delta(g_1 g_3^{-1}) \delta(g_2 g_3^{-1}) \delta(g_1 \tilde{g}_3^{-1}) \delta(g_2 \tilde{g}_3^{-1}) \\ &= \delta(g_3 \tilde{g}_3^{-1})^2 = \delta(\mathbb{1}) \delta(g_3 \tilde{g}_3^{-1}). \end{aligned} \quad (33)$$

The expectation values are calculated according to the trace formula  $\langle O \rangle = \text{tr}(\rho O) / \text{tr} \rho$ . Hence we regularize the density operator by dropping the infinite multiplicative constant and truncating the remaining  $\delta$ -functions in its expansion in term of  $\text{SU}(2)$  representations:

$$\begin{aligned} \rho^J(g_3, \tilde{g}_3) &= \frac{1}{N} \sum_{j_3 \in \mathbb{N}/2}^{j \leq J} (2j_3 + 1) \chi_{j_3}(g_3 \tilde{g}_3^{-1}) \\ &= \frac{1}{N} \sum_{j_3} \sum_{n_3, m_3} (2j_3 + 1) D_{m_3 n_3}^{j_3}(g_3) D_{n_3 m_3}^{j_3}(\tilde{g}_3^{-1}), \end{aligned} \quad (34)$$

where the sum on  $j_3$  is over all half-integers and the  $D_{mn}^j(g)$  are the matrix elements of the (Wigner) matrix representing the group element  $g$  in the representation of spin  $j$  (with  $m$  and  $n$

running by integer step from  $-j$  to  $+j$ ). The normalization constant is determined by the trace condition  $\text{tr } \rho = 1$ , so

$$N = \Delta_J \equiv \sum_{j \in \mathbb{N}/2}^{j \leq J} (2j+1)^2 = \frac{1}{3}(1+2J)(1+J)(3+4J) \sim 8J^3/3. \quad (35)$$

Introducing the following ket notation (as in [10]),

$$\langle g|j, m, n\rangle = \sqrt{2j+1} D_{mn}^j(g), \quad \langle j, m, n|j', m', n'\rangle = \delta_{jj'} \delta_{mm'} \delta_{nn'}, \quad (36)$$

the state induced on the edge  $e_3$  can be written as

$$\rho_3^J = \frac{1}{\Delta_J} \sum_{j_3} \sum_{m_3, n_3} |j_3 m_3 n_3\rangle \langle j_3 m_3 n_3|. \quad (37)$$

For a pure state  $|\Psi\rangle$  the von Neumann entropy of  $\rho_{(3)}^J$  is the measure of entanglement between the edge  $e_3$  and the rest (see e.g. [10]). Hence

$$E(\Psi|3:1, 2) := S(\rho_3^J) = \log \Delta_J \sim 3 \log J + \log(8/3). \quad (38)$$

The result is obviously divergent when  $J \rightarrow \infty$ . We can check that computing the entanglement between the edge  $e_1$  and the rest (edges  $e_2, e_3$ ) gives the same result. Moreover, as a consistency check, we calculate the complementary density matrix  $\rho(g_1, \tilde{g}_1; g_2, \tilde{g}_2)$ ,

$$\rho_{(12)}(g_1, \tilde{g}_1; g_2, \tilde{g}_2) = \int dg_3 \Psi(g_1, g_2, g_3) \Psi(\tilde{g}_1, \tilde{g}_2, g_3) = \delta(G_{12}) \delta(\tilde{G}_{12}) \delta(\tilde{g}_1 g_1^{-1}). \quad (39)$$

The loop  $G_{12} = g_2 g_1^{-1}$  is the analog of an interior loops while  $g_1 g_3^{-1}$  plays the role of the boundary holonomy. Finally, as the reduced state  $\rho^{(12)}$  is the direct product of the pure interior state  $\delta(G_{12}) \delta(\tilde{G}_{12})$  and of the mixed state  $\delta(\tilde{g}_1 g_1^{-1})$ , its entropy is determined only by the latter and we see that  $S(\rho_J^{(12)}) = \log \Delta_J = S(\rho_J^{(3)})$  as expected.

We can further introduce a topological defect along one of the loops of the  $\Theta$ -graph. Let us consider the state

$$\Psi_\theta = \delta_\theta(g_1 g_3^{-1}) \delta(g_2 g_3^{-1}), \quad (40)$$

which satisfies the following constraints, for all  $j \in \mathbb{N}/2$ ,

$$\begin{cases} \chi_j(g_1 g_3^{-1}) \Psi_\theta(g_1, g_2, g_3) = \chi_j(g_1 g_2^{-1}) \Psi_\theta(g_1, g_2, g_3) = \chi_j(\theta) \Psi_\theta(g_1, g_2, g_3), \\ \chi_j(g_2 g_3^{-1}) \Psi_\theta(g_1, g_2, g_3) = (2j+1) \Psi_\theta(g_1, g_2, g_3). \end{cases}$$

Expanding the  $\delta_\theta$ -distribution in  $\text{SU}(2)$  characters, we compute the reduced density matrix for the edge  $e_1$ :

$$\rho_\theta^{(1)}(g_1, \tilde{g}_1) = \sum_{j \in \mathbb{N}/2} \frac{\chi_j^2(\theta)}{2j+1} \chi_j(g_1 \tilde{g}_1^{-1}). \quad (41)$$

After truncation of the sum over representations and proper normalisation, this gives in the ket notation:

$$\rho_\theta^{(1)} = \frac{1}{N(J, \theta)} \sum_{j \leq J} \sum_{m, n} \frac{\chi_j^2(\theta)}{(2j+1)^2} |jmn\rangle \langle jmn|,$$

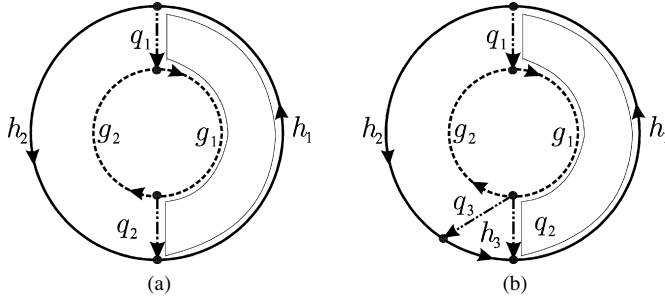


Fig. 4. Interior edges are drawn as broken lines, exterior edges are solid. A boundary loop is indicated by a thin line.

$$N(J, \theta) \equiv \sum_{j \leq J} \chi_j^2(\theta) = \frac{(3 + 4J) \sin \theta - \sin(3 + 4J)\theta}{4 \sin^3 \theta}. \quad (42)$$

For  $\theta = 0$ , we recover the previous reduced density matrix computed for the flat state. This leads to a regularized entanglement as we obtain in the generic case:

$$E(\Psi_\theta | 1 : 2, 3) = \log N(J, \theta) - \frac{1}{N(J, \theta)} \sum_{j \leq J} \chi_j^2(\theta) \log \frac{\chi_j^2(\theta)}{(2j + 1)^2}. \quad (43)$$

This quantity is plotted for two different values of  $j$  on Fig. 2. By hand, we can also compute the reduced density operator  $\rho_\theta^{(2)}$  for the edge  $e_2$ :

$$\begin{aligned} \rho_\theta^{(2)}(g_2, \tilde{g}_2) &= \int dg_1 dg_3 \delta_\theta(g_1 g_3^{-1}) \delta(g_2 g_3^{-1}) \delta_\theta(g_1 g_3^{-1}) \delta(\tilde{g}_2 g_3^{-1}) \\ &= \delta(g_2 \tilde{g}_2^{-1}) \int dG \delta_\theta(G)^2. \end{aligned} \quad (44)$$

Up to a normalisation, it is actually equal to the density matrix  $\rho^{(2)}$  computed above for the flat state,  $\rho_\theta^{(2)}(g_2, \tilde{g}_2) \propto \delta(g_2 \tilde{g}_2^{-1}) = \rho_{(2)}(g_2, \tilde{g}_2)$ . Similarly, we obtain that  $\rho_\theta^{(3)}(g_3, \tilde{g}_3) \propto \delta(g_3 \tilde{g}_3^{-1})$ . That means that imposing a non-trivial holonomy around the single loop  $[e_1, e_2]$  by  $\delta_\theta(g_1 g_2^{-1})$  does not influence the entanglement for the edge  $e_3$  vs.  $[e_1, e_2]$  or for the edge  $e_2$  vs.  $[e_1, e_3]$ .

#### 4.2. Further examples

A simple graph that is represented on Fig. 4(a) has an equal number of all three types of edges,  $E_{\text{int}} = E_{\text{b}} = E_{\text{ext}} = 2$  and allows a simple inside/outside distinction. The holonomies along the internal, boundary and external edges are denoted by  $g$ ,  $q$ , and  $h$ , respectively. It has two internal and two external vertices, so the three independent loops are produced by one internal, one external, and one boundary loop. The loops are marked on Fig. 4(a). The flat wave functional is given by (taking into account the orientation of the edges)

$$\Psi_0 = \delta(g_1 g_2) \delta(h_1 h_2) \delta(g_1 q_2 h_1 q_1). \quad (45)$$

Computing the reduced density matrix for the two interior edges gives:

$$\rho_{\text{int}}^{\Psi_0}(g, \tilde{g}) = \delta(G) \delta(\tilde{G}) \delta(\tilde{g}_1 g_1^{-1}), \quad (46)$$

where  $G = g_1 g_2$  is the holonomy along the interior loop. This interior density matrix decomposes into a direct product of a pure state on the interior loop and the mixed state  $\delta(\tilde{g}_1 g_1^{-1})$ . According



to the previous calculations, the regularized entropy is simply

$$S(\rho_{\text{int}}^{\psi_0}) = \log \Delta_J. \quad (47)$$

It is easy to check that a different choice of loop or increase in their number only results in the appearance of additional  $\delta(\mathbb{1})$  factors that do not alter the regularized entropy.

Let us look at the possible addition of a new boundary edge without changing the number of boundary vertices as in Fig. 4(b). The new boundary edge actually does not create a new boundary loop but simply a new exterior loop (since it does not involve any interior edge). Thus it should not contribute to the entanglement. Indeed the new flat state is:

$$\Psi(g, q, h) = \delta(g_1 g_2) \delta(h_1 h_2 h_3) \delta(g_1 q_2 h_1 q_1) \delta(q_3 h_3 q_2^{-1}). \quad (48)$$

As expected, it results in the same reduced density matrix for the interior,  $\rho_{\Psi}^{\text{int}} = \rho_{\psi_0}^{\text{int}}$ , thus also  $S(\rho_{\Psi}^{\text{int}}) = S(\rho_{\psi_0}^{\text{int}})$ .

Coming back to the original graph in Fig. 4(a), we impose a non-trivial holonomy around the boundary loop:

$$\Psi_{\theta} = \delta(g_1 g_2) \delta(h_1 h_2) \delta_{\theta}(g_1 q_2 h_1 q_1). \quad (49)$$

The interior density operator is

$$\rho_{\text{int}}^{\Psi_{\theta}}(g, \tilde{g}) = \delta(G) \delta(\tilde{G}) \delta_{\theta}(\tilde{g}_1 g_1^{-1}), \quad (50)$$

and it leads to a renormalized entropy equals to  $\bar{E}(\theta)$  as we computed in the previous section (see Result 3).

## 5. Conclusions

We computed the entropy for a bounded region on a physical state of BF theory. Indeed, being solvable and lacking local degrees of freedom, BF theory allows for explicit calculations and a precise analysis of the relationship between boundaries and degrees of freedom. Looking at physical spin network states, we showed that the entanglement between the two regions for an arbitrary bipartite splitting of the spin network only depends on the structure of the boundary: the entropy simply scales with the size of the boundary. More precisely, we proved that the entanglement grows with the number of boundary vertices,  $E = (V_b - 1) \log \Delta_J$ . The factor  $\log \Delta_J \sim 3 \log J$  actually diverges as the cut-off regulator  $J$  is sent to infinity. Such divergences naturally occur when the gauge group of the theory is not a discrete group but a Lie group: for  $SU(2)$  the number of modes that can be excited increases as  $J^3$  with the cut-off  $J$  in representation space.

The area dependence of the regularized (ground state) entropy is already well-established in quantum field theory and we showed that the same result holds for a pure BF theory as expected. However, the fact that pure BF theory is a topological field theory allows to control the introduction of local degrees of freedom through the insertion of topological defects. We showed that the introduction of such topological defects does not affect the regularized entropy at leading order. Moreover, we computed the *finite* entropy difference due to such particle-like defects. We proved that defects only affects the entropy counting when they are located on the boundary and not when they are introduced in the bulk. We showed how the entropy difference is related to the mass of the particles: it actually does not seem to depend on the defect angle except for some specific values where we get resonances. This is a new and unexpected result which calls

for a physical interpretation. This entropy difference between configurations with particles or no particle on the boundary should be related to the process of entropy creation during black hole evaporation (or particles falling through a horizon). However, in order to work out explicitly such a relation, we would need to extend the previous computations from BF theory to gravity.

Technically, we developed the necessary mathematical tools required to analyze the graph structure of a flat distributional spin network state and to regularize the resulting wave functional and entropy calculations. These tools are also relevant to entanglement calculations for the spin systems used for topological quantum computation such as the Kitaev model [5,6].

We hope to apply these techniques to study the precise relation between gauge breaking and entropy on one hand, and to compute the entanglement for a more general class of spin network states relevant for loop quantum gravity [16].

## Appendix A. Computing the renormalized entanglement

Here we prove that the renormalized entanglement  $\bar{E}(\theta)$ , as introduced in Section 3.3,

$$\bar{E}_J[\theta] = \lim_{J \rightarrow \infty} S(\rho_{\text{int}}^{\theta J}) - S(\rho_{\text{int}}^J) = \lim_{J \rightarrow \infty} \log \frac{N_J(\theta)}{\Delta_J} - \frac{1}{N_J(\theta)} \sum_{j \leq J} \chi_j(\theta)^2 \log \frac{\chi_j(\theta)^2}{(2j+1)^2},$$

converges for any value of the angle  $\theta$ . Because of the symmetry  $\theta \leftrightarrow \pi - \theta$ , we can restrict ourselves to  $0 < \theta \leq \pi/2$ . Since

$$N_J(\theta) \sim \frac{J}{\sin^2 \theta}, \quad \Delta_J \sim 8J^3/3, \quad (\text{A.1})$$

it follows that the limit exists if asymptotically

$$S(\rho_{\text{int}}^{\theta J}) \sim 3 \log J + f(\theta) + \mathcal{O}(1/J), \quad (\text{A.2})$$

for some function  $f(\theta)$ .

We have to consider two different cases, depending on whether the class angle is a rational or irrational fraction of  $\pi$ . In the first case the class angle is  $\theta = p/q\pi$ , with relatively prime  $p, q \in \mathbb{N}$ , and there are only finitely many different values that  $\sin^2 n\theta$  takes. It is periodic with period  $K_\theta = q$ , and

$$\sum_{l=0}^q \sin^2(lp\pi/q) = q/2. \quad (\text{A.3})$$

To simplify the notation we further take  $J = kq$ ,  $k \in \mathbb{N}$ , and consider the limit  $k \rightarrow \infty$ .

As a result, the entropy

$$S(\rho_{\text{int}}^{\theta J}) \sim \log J - 2 \log \sin \theta - \frac{1}{J} \sum_{n=0}^{2kq+1} \sin^2 n\theta \log \frac{\sin^2 n\theta}{n^2 \sin^2 \theta}, \quad (\text{A.4})$$

becomes

$$\begin{aligned} S(\rho_{\text{int}}^{\theta J}) \sim \log J + \frac{1}{kq} \sum_{j=0}^{2k} \sum_{l=0}^{q-1} (2 \sin^2(l\theta) \log(jq+l) - \sin^2(l\theta) \log[\sin^2(l\theta)]) \\ + f(\theta) + \mathcal{O}(1/J), \end{aligned} \quad (\text{A.5})$$

where we expressed the summation index  $n$  by  $n = jq + l$ . The entropy finally reduces to

$$\begin{aligned} S(\rho_{\text{int}}^{\theta J}) &\sim \log J + \frac{2}{kq} \frac{q}{2} (\log[(2k)!] + 2k \log q) + f(\theta) + \mathcal{O}(1/J) \\ &= 3 \log J + \tilde{f}(\theta) + \mathcal{O}(1/J), \end{aligned} \quad (\text{A.6})$$

which establishes the claim for  $\theta = p/q\pi$ ,  $p, q \in \mathbb{N}$ .

For a generic value of  $\theta$  we establish the limit by using the Euler–Maclauren integration formula for sums. The asymptotic behavior of the entropy is

$$\begin{aligned} S(\rho_{\text{int}}^{\theta J}) &\sim \log J + \frac{1}{J} \left( 2 \int_0^{2J} dn \sin^2(n\theta) \log n - \int_0^{2J} dn \sin^2(n\theta) \log \sin^2(n\theta) \right) \\ &\quad + f(\theta) + \mathcal{O}(1/J). \end{aligned} \quad (\text{A.7})$$

Using the known integral  $\int_0^\pi dx \sin^2 x \log \sin^2 x = \pi(\frac{1}{2} - \log 2)$ , the second term becomes

$$\frac{1}{J} \int_0^{2J} dn \sin^2(n\theta) \log \sin^2(n\theta) \sim 1 - 2 \log 2. \quad (\text{A.8})$$

The first integral has a closed form that involves sine and cosine integral functions, but the relevant part is simply

$$\frac{1}{J} \int_0^{2J} dn \sin^2(n\theta) \log n \sim -2 + 2 \log 2 + 2 \log J + \dots, \quad (\text{A.9})$$

hence the limit exists for all  $\theta$  and for the irrational fractions of  $\pi$

$$\bar{E}_A(\Psi_\theta) = -3 + \log 6. \quad (\text{A.10})$$

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